equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \ldots, y_n$, it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{16}$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (16) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $y_1, \ldots, y_n$ of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

To find a particular solution $Y(t)$ in Eq. (16), the methods of undetermined coefficients and variation of parameters are again available. They are discussed and illustrated in Sections 4.3 and 4.4, respectively.

The method of reduction of order (Section 3.4) also applies to nth order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n-1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6, determine intervals in which solutions are sure to exist.

1. $y^{(4)} + 4y''' + 3y = t$
2. $ty''' + (\sin t)y'' + 3y = \cos t$
3. $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$
4. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
5. $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$
6. $(x^2 - 4)y^{(6)} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10, determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

7. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
8. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
9. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$
10. $f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

11. $y''' + y' = 0; \quad 1, \quad \cos t, \quad \sin t$
12. $y^{(4)} + y'' = 0; \quad 1, \quad t, \quad \cos t, \quad \sin t$
13. $y''' + 2y'' - y' - 2y = 0; \quad e^t, \quad e^{-t}, \quad e^{-2t}$
14. $y^{(4)} + 2y''' + y'' = 0; \quad 1, \quad t, \quad e^{-t}, \quad te^{-t}$
15. $xy''' - y'' = 0; \quad 1, \quad x, \quad x^3$
16. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \quad x, \quad x^2, \quad 1/x$
17. Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?
18. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

PROBLEMS

In each of Problems 1 through 8, determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{(4)} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y''' - y' = 2\sin t$
5. $y^{(4)} - 4y'' = t^2 + e^t$
6. $y^{(4)} + 2y'' + y = 3 + \cos 2t$
7. $y^{(6)} + y''' = t$
8. $y^{(4)} + y''' = \sin 2t$

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t;\quad y(0) = y'(0) = 0,\quad y''(0) = 1$
10. $y^{(4)} + 2y'' + y = 3t + 4;\quad y(0) = y'(0) = 0,\quad y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t;\quad y(0) = 1,\quad y'(0) = -\frac{1}{4},\quad y''(0) = -\frac{3}{2}$
12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t};\quad y(0) = 3,\quad y'(0) = 0,$ $y''(0) = -1,\quad y'''(0) = 2$

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y''' - 2y'' + y' = t^3 + 2e^t$
14. $y''' - y' = te^{-t} + 2\cos t$
15. $y^{(4)} - 2y'' + y = e^t + \sin t$
16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$
17. $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$
19. Consider the nonhomogeneous nth order linear differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = g(t), \tag{i}$$

where $a_0, \dots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces Eq. (i) to the form

$$k_0u^{(n)} + k_1u^{(n-1)} + \cdots + k_nu = b_0t^m + \cdots + b_m, \tag{ii}$$

where $k_0, \dots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 20 through 22, we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for d/dt. Then, for example, e^{-t} is a solution of $(D + 1)y = 0$; the differential operator $D + 1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t}. In the same way, $D^2 + 4$ is an annihilator of $\sin 2t$ or $\cos 2t$, $(D - 3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

In each of Problems 7 through 10, follow the procedure illustrated in Example 4 to determine the indicated roots of the given complex number.

7. $1^{1/3}$
8. $(1 - i)^{1/2}$
9. $1^{1/4}$
10. $[2(\cos \pi/3 + i \sin \pi/3)]^{1/2}$

In each of Problems 11 through 28, find the general solution of the given differential equation.

11. $y''' - y'' - y' + y = 0$
12. $y''' - 3y'' + 3y' - y = 0$
13. $2y''' - 4y'' - 2y' + 4y = 0$
14. $y^{(4)} - 4y''' + 4y'' = 0$
15. $y^{(6)} + y = 0$
16. $y^{(4)} - 5y'' + 4y = 0$
17. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$
18. $y^{(6)} - y'' = 0$
19. $y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0$
20. $y^{(4)} - 8y' = 0$
21. $y^{(8)} + 8y^{(4)} + 16y = 0$
22. $y^{(4)} + 2y'' + y = 0$
23. $y''' - 5y'' + 3y' + y = 0$
24. $y''' + 5y'' + 6y' + 2y = 0$
25. $18y''' + 21y'' + 14y' + 4y = 0$
26. $y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$
27. $12y^{(4)} + 31y''' + 75y'' + 37y' + 5y = 0$
28. $y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0$

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

29. $y''' + y' = 0$; $\quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$
30. $y^{(4)} + y = 0$; $\quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$
31. $y^{(4)} - 4y''' + 4y'' = 0$; $\quad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$
32. $y''' - y'' + y' - y = 0$; $\quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$
33. $2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0$; $\quad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
34. $4y''' + y' + 5y = 0$; $\quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
35. $6y''' + 5y'' + y' = 0$; $\quad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0$
36. $y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0$; $\quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0,$ $y'''(0) = 3$
37. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

38. Consider the equation $y^{(4)} - y = 0$.

(a) Use Abel's formula [Problem 20(d) of Section 4.1] to find the Wronskian of a fundamental set of solutions of the given equation.

(b) Determine the Wronskian of the solutions $e^t, e^{-t}, \cos t$, and $\sin t$.

(c) Determine the Wronskian of the solutions $\cosh t, \sinh t, \cos t$, and $\sin t$.

39. Consider the spring–mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.

(a) Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1. \qquad \text{(i)}$$